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# A LOWER BOUND ON CONNECTIVITIES OF MATROID BASE GRAPHS

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The connectivity of a graph $G$ and the corank of a matroid $M$ are denoted by $\kappa(G)$ and $\rho$, respectively. It is shown that if a graph $G$ is the base graph of a simple matroid $M$, then $\kappa(G) \geq 2\rho$ and the lower bound of $2\rho$ is best possible.

## 1. Introduction

Holzmann and Harary [4] showed that for every edge of a matroid base graph there is a Hamilton cycle containing it and another Hamilton cycle excluding it. Alspach and Liu [1] proved that the base graph of a simple matroid is Hamilton-connected and edge-pancyclic. Maurer [7] discussed the properties of subgraphs of matroid base graphs. Holzmann et al. [5] proved that two matroids are equivalent if and only if their base graphs are isomorphic. Donald et al. [3] gave the characterization of base graphs of complete matroids. Liu [6] proved that the base graph of any matroid $M$ without loops is $\rho$-connected where $\rho$ is the corank of $M$. This paper is the continuation of [6]. It is proved that the base graph of any simple matroid with corank $\rho$ is $2\rho$-connected, and the lower bound cannot be improved.

For convenience we denote the union of $S$ and $S'$ by $S + S'$. A matroid $M$ is a finite set $E$ together with a nonempty collection $B$ of subsets of $E$ such that the following condition is satisfied: For any $b$, $b' \in B$ and $e \in b - b'$, there exists $e' \in b' - b$ such that $b - e + e' \in B$. Write $M = (E, B)$. Each member of $B$ is called a base of $M$. Any subset of a base is called independent. A subset of $E$ which is not independent is called dependent. A circuit $M$ is a minimal dependent set. Let $b$ be a base of $M$. For any $e \notin b$, $b + e$ contains an unique circuit [8], which is called a basic circuit of $M$. The rank $r$ and the corank $\rho$ of a matroid $M$ are the cardinal of a base and the number of basic circuits with respect to one base $b$, respectively. Clearly, $r + \rho = |E|$. For a fixed element $e$ of $E$, $M \Delta e$ and $M \cdot e$ denote the matroid obtained from $M$ by deleting and contracting the element $e$, respectively, that is, $M \Delta e = (E', B')$ and $M \cdot e = (E', B'')$ are defined by

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$E' = E - e$, $B' = \{b\colon\ b \in B \text{ and } e \notin b\}$ and $B'' = \{b - e\colon\ b \in B \text{ and } e \in b\}$. A loop of $M$ is an element of $E$ which is a circuit. A matroid without loops and 2-circuits is called a simple matroid. Other matroid terminology in this paper can be found in [8].

The notation $V(G)$ and $E(G)$ will be used for the vertex set and edge set of a graph $G$, respectively. The base graph of a matroid $M = (E, B)$ is the graph $G$ such that $V(G) = B$ and $E(G) = \{bb'\colon\ b, b' \in B \text{ and } |b' - b| = 1\}$, where the same notation is used for the vertex of $G$ and the base of $M$. If $e \in b$, then we also say that the vertex $b$ of $G$ contains $e$.

If $A \subseteq V(G)$, then $G[A]$ denotes the induced subgraph of $G$ by $A$. Set $G - A = G[V(G) - A]$. The connectivity $\kappa(G)$ of a graph $G$ is the minimum $k$ for which there is a subset $V'$ of $V(G)$ such that $|V'| = k$ and $G - V'$ is disconnected. A walk in a graph $G$ is an alternating sequence of vertices and edges

$$P = (v_1, e_1, v_2, \ldots, v_{k-1}, e_{k-1}, v_k),$$

such that $e_i$ is an edge joining $v_i$ and $v_{i+1}$. For simplicity,

$$P = (v_1, v_2, \ldots, v_k)$$

will be written thereby implying the occurrence of the edges. If $Q = (v_k, v_{k+1}, \ldots, v_n)$ is a walk, then

$$P + Q = (v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n).$$

If $P = (v_1, v_2, \ldots, v_k)$ is a walk in which all the vertices are distinct, then $P$ is called a path. Let $S$ and $S'$ be two subsets of $V(G)$. If one end vertex of a path $P$ is in $S$ and the other is in $S'$, we say that path $P$ joins $S$ to $S'$.

## 2. The connectivity theorem

In the following we denote the base graph of $M$, $M \Delta e$ and $M \cdot e$ by $G$, $G'$ and $G''$, respectively.

**Lemma 1** ([4]). *Let $M = (E, B)$ be any matroid and $e \in E$. Then $G' = G[V']$ and $G'' = G[V'']$, where $V' = V(G) - V''$ and $V'' = \{b\colon\ b \in V(G) \text{ and } e \in b\}$.*

If a graph has a Hamilton cycle, then it is 2-connected. From [4] we obtain the following result.

**Lemma 2.** *Let $G$ be the base graph of a matroid with more than two vertices. Then $G$ is 2-connected.*

**Lemma 3** ([2]). *A graph $G$ with $n \geq k + 1$ vertices is $k$-connected if and only if for any two subsets $S$ and $S'$ of $V(G)$ with $|S| = |S'| = k$, there are $k$ disjoint paths in $G$ joining $S$ to $S'$.*

**Lemma 4** ([2]). *A graph $G$ with $n \geq k+1$ vertices is $k$-connected if and only if any two distinct vertices of $G$ are joined by at least $k$ internally disjoint paths.*

**Lemma 5.** *Let $G$ be the base graph of a simple matroid $M=(E, B)$. If $b$ and $b'$ are two bases of $M$ and $e \in E-(b+b')$, then there exist two internally disjoint paths in $G$ joining $b$ and $b'$ each interior vertex of which contains $e$.*

**Proof.** Let $b+e$ contain the circuit $C$ in $M$. Since $M$ is simple, there are two elements $e_1$ and $e_2$ of $C$ such that $e_1, e_2, \neq e$. Set $b_1=b+e-e_1$ and $b_2=b+e-e_2$. Then $bb_1$, $bb_2 \in E(G)$. Similarly, there exist $b_1'=b'+e-e_1'$ and $b_2'=b'+e-e_2'$ such that $b'b_1'$, $b'b_2' \in E(G)$. By Lemma 1, $b_1$, $b_2$, $b_1'$ and $b_2'$ are vertices of $G''$. We distinguish the following two cases:

*Case* 1. $b_1$, $b_2$, $b_1'$ and $b_2'$ are four distinct vertices

Let the corank of $M$ be $\rho$. Since the element $e \in E-(b+b')$, it follows that $\rho \geq 2$. The corank of $M \cdot e$ is also $\rho$. Since $M$ is simple, $M \cdot e$ has no loops. By Lemma 2 $G''$ is 2-connected. Thus, by Lemma 3, there exist two disjoint paths $Q_1$ and $Q_2$ in $G''$ joining $\{b_1, b_2\}$ to $\{b_1', b_2'\}$. Without loss of generality, assume that $Q_i$ joins $b_i$ and $b_i'$, $i=1,2$. Set

$$P_i=(b, b_i)+Q_i+(b_i', b'), \quad i=1,2.$$

Then $P_1$ and $P_2$ are as required.

*Case* 2. $\{b_1, b_2\} \cap \{b_1', b_2'\} \neq \emptyset$

If $\{b_1, b_2\}=\{b_1', b_2'\}$, assume that $b_i=b_i'$, $i=1, 2$. Then

$$P_i=(b, b_i, b'), \quad i=1, 2,$$

are as required. Otherwise, without loss of generality, assume that $b_1=b_1'$ and $b_2 \neq b_2'$. Since $G''$ is 2-connected, $G''-b_1$ is connected. So there exists a path $Q$ in $G''-b_1$ joining $b_2$ and $b_2'$. Hence

$$P_1=(b, b_1, b') \quad \text{and} \quad P_2=(b, b_2)+Q+(b_2', b')$$

are as required. □

**Lemma 6.** *Let $G$ be the base graph of a simple matroid $M=(E, B)$ and $b, b' \in B$ such that $|b-b'|=2$. Then there exist at least four internally disjoint paths in $G$ joining $b$ and $b'$.*

**Proof.** Let $b_1=b+e_1'-e_1$ and $b'=b_1+e_2'-e_2$. Let $b+e_i'$ contain the circuit $C_i$, $i=1,2$. We claim that $b_2=b+e_2'-e_2 \in B$. Otherwise, we have $e_2 \notin C_2$. Since $b'$ contains no circuits, it follows that $e_1 \in C_2$. By the circuit axiom of matroids $(C_1+C_2)-e_1 \subseteq b'$ contains a circuit, a contradiction. Thus $b'=b_2+e_1'-e_1$. Set

$$P_i=(b, b_i, b'), \quad i=1, 2.$$

In order to construct $P_3$ and $P_4$ we consider the following three cases.

*Case* 1. $e_1, e_2 \in C_1 \cap C_2$
Set $b_3 = b + e_2' - e_1$ and $b_4 = b + e_1' - e_2$. Set

$$P_3 = (b, b_3, b') \quad \text{and} \quad P_4 = (b, b_4, b').$$

*Case* 2. Without loss of generality, assume that $e_1 \in C_1 \cap C_2$ and $e_2 \in C_2 - C_1$. Since $M$ is simple, there exists an element $e$ of $C_1$ such that $e \neq e_1', e_1$. Clearly, $e \neq e_2$. Set $b_3 = b + e_1' - e$. Let $b_3 + e_2'$ contain the circuit $C_3$. If $e \notin C_1 \cap C_2$, then $C_3 = C_2$. So $e_2 \in C_3$. If $e \in C_1 \cap C_2$, by the circuit axiom of matroids, for $e_2 \in C_2 - C_1$ there is a circuit $C \subseteq (C_1 + C_2) - e$ such that $e_2 \in C$. Since $(C_1 + C_2) - e \subseteq b_3 + e_2'$, from the uniqueness of basic circuits, $C_3 = C$. Thus $e_2 \in C_3$. Set $b_4 = b_3 + e_2' - e_2$. Set

$$P_3 = (b, b_3, b_4, b'),$$

and set

$$P_4 = (b, b_5, b'),$$

where $b_5 = b + e_2' - e_1$.

*Case* 3. $e_1, e_2 \notin C_1 \cap C_2$
As in Case 2, set

$$P_3 = (b, b_3, b_4, b'),$$

where $b_3 = b + e_1' - e$ and $b_4 = b_3 + e_2' - e_2$. Since $e_1 \in C_1 - C_2$, similarly, there are bases $b_5 = b + e_2' - e''$ and $b_6 = b_5 + e_1' - e_1$, where $e'' \in C_2$ and $e'' \neq e_2', e_2$. Set

$$P_4 = (b, b_5, b_6, b').$$

It is easy to see that $P_1$, $P_2$, $P_3$ and $P_4$ are internally disjoint paths in $G$ joining $b$ and $b'$. □

By the definition of a matroid base graph the following result holds.

**Lemma 7.** *Let $G$ be the base graph of a matroid $M$ with the corank $\rho$ and let $m$ be the size of the smallest circuit of $M$. Then the minimum degree of vertices in $G$ is at least $(m-1)\rho$.*

**Theorem.** *Let $G$ be the base graph of a simple matroid $M = (E, B)$ with corank $\rho$. Then $\kappa(G) \geq 2\rho$.*

**Proof.** By Lemma 7, $G$ has at least $2\rho + 1$ vertices. By Lemma 4, in order to prove the theorem it suffices to prove that for any two distinct vertices $b$ and $b'$ of $G$ there exist $2\rho$ internally disjoint paths in $G$ joining them. When $\rho = 0$, the theorem is trivial. So suppose that $\rho > 0$. We shall prove the above result by induction on $\rho$. When $\rho=1$, $G$ is a complete graph with $n\geq 3$ vertices. Thus, $\kappa(G)\geq 2\rho=2$. So we suppose that the result holds for $\rho-1$. We are going to prove that it also holds for $\rho>1$. Let $|b-b'|=d$. We distinguish between the following two cases:

*Case* 1. $\rho>d$

Since $\rho=|E|-|b|$ and $\rho>d$, it follows that there is an element $e\in E-(b+b')$. The corank of $M\,\Delta\, e$ is $\rho-1$ and $b,\ b'\in V(G')$. By the induction hypothesis there exist $2\rho-2$ internally disjoint paths in $G'$ joining $b$ and $b'$ each vertex of which does not contain $e$. By Lemma 5 there exist two internally disjoint paths in $G$ joining $b$ and $b'$ each interior vertex of which contains $e$. So the result is true.

*Case* 2. $\rho=d$

By the above hypothesis $d=\rho\geq 2$. When $d=2$, from Lemma 6, the result is true. When $d>2$, to prove that there exist $2\rho$ internally disjoint paths in $G$ joining $b$ and $b'$ it sufficies to prove that for any $(2\rho-1)$-subset $A$ of $V(G)$ and $b,\ b'\notin A$, there exists a path in $G-A$ joining $b$ and $b'$. Set $b'-b=\{e_1',\ e_2',\ldots,e_d'\}$. Let $b+e_i'$ contain the circuit $C_i$. Then there must exist an element $e_i\in C_i$ such that $e_i\in b-b'$. So $b_i=b+e_i'-e_i\in B$. Similarly, for any $e_{i+1}'\in b'-b_i$, there exists $e_{i+1}''\in b_i-b'$ such that $b_i'=b_i+e_{i+1}'-e_{i+1}''$ is in $B(1\leq i\leq d,\ e_{d+1}'=e_1')$. Obviously, $b_1,b_2,\ldots,b_d,b_1',b_2',\ldots,b_d'$ are $2d$ distinct vertices of $G$. Set

$$B_1=\{b_1,b_2,\ldots,b_d,b_1',b_2',\ldots,b_d'\}.$$

Since $|b-b_i|=1$, $|b-b_i'|=2$ and $|b-b'|=d>2$, it follows that $b'\notin B_1$. Let $A\subseteq V(G)$ contain $2\rho-1$ elements and $b,b'\notin A$. Then $B_1-A\neq\emptyset$. Thus for some element $b_i\in B_1-A$, we have $bb_i\in E(G)$. Since $|b'-b_i|<d=\rho$, by Case 1, it follows that there exist $2\rho$ internally disjoint paths in $G$ joining $b_i$ and $b'$. Since $|A|=2\rho-1$, there exists a path $Q$ in $G-A$ joining $b_i$ and $b'$. If $Q$ passes through $b$, then there exists a subpath $P$ of $Q$ joining $b$ and $b'$. Otherwise, $P=(b,b_i)+Q$ is as desired. If $b_i\in A$ for all $i$, then there exists an element $b_j'\in B_1-A$. Since $|b-b_j'|=2<\rho$, by Case 1, there exist $2\rho$ internally disjoint paths in $G$ joining $b$ and $b_j'$. So there exists a path $Q_1$ in $G-A$ joining them. Similarly, since $|b'-b_j'|<d=\rho$, by Case 1, there exists a path $Q_2$ in $G-A$ joining $b'$ and $b_j'$. Thus the walk $Q_1+Q_2$ joins $b$ and $b'$ in $G-A$. This completes the proof. □

**Corollary.** *Let $G$ be the base graph of a matroid $M=(E,B)$ with corank $\rho$. If each circuit of $M$ is a 3-circuit, then $\kappa(G)=2\rho$.*

**Proof.** By the above theorem $\kappa(G)\geq 2\rho$. Since each circuit of $M$ is a 3-circuit, by Lemma 7, $G$ is regular of degree $2\rho$. Thus, $\kappa(G)\leq 2\rho$. So $\kappa(G)=2\rho$. □

There are matroid base graphs, such as the complete matroid base graph $B_{m,2}$, whose connectivities are $2\rho$. In the above sense the lower bound in this paper is best possible.

If a matroid is not simple, then the theorem in this paper is not true. There are base graphs of no simple matroids, such as the $\rho$-cube, whose connectivities are $\rho$ [6].

For the general case I give the following two conjectures.

**Conjecture 1.** *If G is the base graph of a matroid with corank $\rho$ each circuit of which has cardinal not less than $k+1$, then $\kappa(G) \geq k\rho$.*

**Conjecture 2.** *Let G be the base graph of any matroid. Then $\kappa(G) = \lambda(G)$ where $\lambda(G)$ and $\delta(G)$ are the edge-connectivity and the minimum degree of vertices of G.*

Clearly, Conjecture 2 implies Conjecture 1. Possibly these conjectures can be proven by a method similar to that of this paper.

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